

# THE WEIBULL SHAPE PARAMETER AND THE HEALTHY LIFE YEARS LOST

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## Abstract

The Weibull model and the cumulative hazard can express the additive process which disabilities and diseases cause to the human organism during the life span. In this paper we further analytically derive a more general model of survival-mortality in which we estimate a parameter related to the Healthy Life Years Lost (HLYL) leading to the Weibull model and the corresponding shape parameter as a specific case. We have also demonstrated that the estimates for the general HLYL parameter we have proposed, provide results similar to those of the World Health Organization for the Healthy Life Expectancy (HALE) and the corresponding HLYL estimates. An analytic derivation of the mathematical formulas is presented along with an easy to apply Excel program. This program is an extension of the classical life table including more columns to estimate the cumulative mortality, the average mortality, the person life years lost and finally the HLYL parameter  $b_x$ . Evenmore, a further extension of the Excel program based on the Sullivan method provides estimates of the Healthy Life Expectancy at every year of the lifespan for five different types of estimates that are the Direct, WHO, Eurostat, Equal and Other. Estimates for Czechia are presented whereas the program is ready for applications in other countries.

**Key words:** Healthy life expectancy, healthy life years lost, Weibull, WHO, HALE, Eurostat, Sullivan.

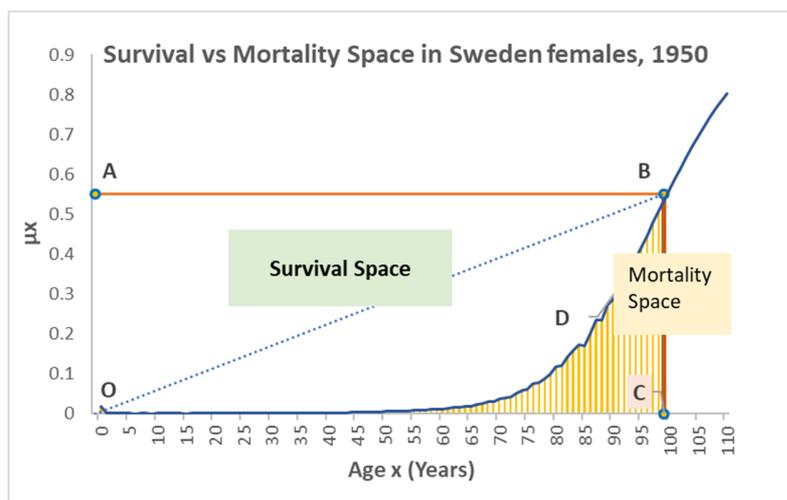
**JEL Code:** I15, I18, J1

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## 1. The HLYL Estimation Method

Our methodology Skiadas and Skiadas (2014, 2015, 2018 a,b,c, 2019a,b) was based on a geometric approach from the following graph of mortality spaces where both mortality and survival are presented as appropriate areas of this graph.

**Fig. 1. Survival vs Mortality space graph**



Source: author's calculations

The usual form to express mortality  $\mu_x$  in a population at age  $x$  is by estimating the fraction Death( $D_x$ )/Population( $P_x$ ) that is  $\mu_x = D_x/P_x$ . As in the following we will use the Life Tables provided from the Human Mortality Database we will use the term  $m_x$  of these tables instead of  $\mu_x$ . The above graph using data from Sweden 1950 females from the HMD is formulated with  $\mu_x$  as the blue exponential curve. The main forms of Life Tables start with  $\mu_x$  and in the following estimate the survival forms of the population. This methodology leads to the estimation of a probability measure termed as life expectancy at age  $x$  or life expectancy at birth when considering the total life time. There are several differences between the graph with the survival space above and the survival curves methodology. First of all, the vertical axis in the Survival-Mortality Space (SMS) diagram is the probability  $\mu_x$ . Instead in the survival diagram the vertical axis represent population (usually it starts from 100.000 in most life tables and gradually slow down until the end). By the SMS diagram we have probability spaces for both survival and mortality. For the age  $x$  the total space is (ABCOA) in the SMS diagrams that is  $(OA) \cdot (BC) = x \mu_x$ . The mortality space is the sum  $S(\mu_x)$  and survival space is  $(x\mu_x - S(\mu_x))$ . Accordingly, the important measure of the Health State is simply the fraction  $(ABDOA)/(BCODB)$ . Simpler is to prefer the fraction  $(ABCOA)/(BCODB) = x\mu_x / S(\mu_x)$  that can be estimated from  $\mu_x$  for every age  $x$  of the population.

In modeling the healthy life years lost to disability some important issues should be realized. Mortality expressed by  $\mu_x$  is important for modeling disability but more important is the

cumulative mortality  $S \mu_x$  which, as an additive process, is more convenient for the estimation of the healthy life years lived with disability and the deterioration process causing deaths. The estimates for this type of mortality are included in the term  $b_x S \mu_x$ .

Our approach in previous publications (Skiadas and Skiadas (2018a,b,c, 2019)) was to set a time varying fraction  $b_x$  for Health/Mortality of the form:

$$b_x = \frac{x\mu_x}{\int_0^x \mu_s ds} \quad (1)$$

This formula is immediately provided from figure 1 by considering the fraction:

$$b_x = \frac{\text{Total Space}}{\text{Mortality Space}} = \frac{OABCO}{ODBCO} = \frac{x\mu_x}{\int_0^x \mu_s ds}$$

It should be noted that an alternative approach is given by:

$$b_x = \frac{\text{Survival Space}}{\text{Mortality Space}} = \frac{OABCO - ODBCO}{ODBCO} = \frac{x\mu_x}{\int_0^x \mu_s ds} - 1$$

In the latter case the estimated fraction  $b_x$  is smaller by one from the previous case. It remains to the applications stage to decide for the most appropriate. So far the Total Space approach is simpler and gave good results.

The main hypothesis is that the population involved in the deterioration process is a fraction of the total population determined by the level of mortality  $\mu_x$  at age  $x$ . Accordingly the mortality process will have two alternatives expressed by the simple equation:

$$x\mu_x = b_x \int_0^x \mu_s ds \approx b_x \sum_0^x \mu_x \quad (2)$$

Where  $x\mu_x$  is the incoming part related to the disability of the living population and the second part is the outgoing part that is summed to the mortality for the period from 0 to age  $x$ . The parameter  $b_x$  is a corresponding adding to express the rate of healthy life lost to disability. The applications verify that the maximum values for  $b=b_{max}$  are compatible to the estimates of the WHO for several countries. Evenmore, our estimates expressing the values for  $b_x$  in all the life time are of particularly importance in the studies related to the Health Expenditure estimation.

Some important properties of the last formula are given below:

First we can formulate the Survival Probability  $S(t)$

$$S(t) = \exp\left(-\int_0^x \mu_s ds\right) = \exp\left(-\frac{x\mu_x}{b_x}\right) \approx \exp\left(-\sum_0^x \mu_x\right) \quad (3)$$

$$S(t) \approx \exp\left(-\sum_0^x \mu_x\right) = \exp(-\mu_0) \exp(-\mu_1) \exp(-\mu_2) \dots \exp(-\mu_x)$$

Next we can differentiate (2) to obtain

$$(x\mu_x)' = b'_x \int_0^x \mu_s ds + b_x \mu_x \quad (4)$$

For a constant  $b$  we have  $b'_x = 0$  and  $x\mu'_x + \mu_x = b\mu_x$

It follows  $x\mu'_x = (b - 1)\mu_x$  and rearranging  $\frac{\mu'_x}{\mu_x} = \frac{b-1}{x}$

Solving the differential equation  $\ln(\mu_x) = \ln(c) + (b - 1) \ln x$

Where  $c$  is a constant of integration. Finally  $\mu_x = cx^{b-1}$

By setting  $c=\lambda b$  the hazard function or the generating function of the Weibull appear

$$\mu_x = \lambda bx^{b-1} \quad (5)$$

And the cumulative hazard of the Weibull is

$$\Lambda(x) = \lambda x^b = \frac{x\mu_x}{b} = \int_0^x \mu_s ds \quad (6)$$

This is to verify the formula for the survival probability (3) presented earlier.

As we already have presented in previous studies (Skiadas and Skiadas (2018a,b,c, 2019)),  $b_x$  can be estimated directly from the life table data. The estimates with the direct method are close to the WHO estimates. The results verify that both methods approach well between each other. Of course the Direct method, based on only the life tables can used in all the time periods as far as life tables exist.

## 2. Program for the Estimates

We have developed an Excel program for the Direct Estimates of  $b_x$  which is provided free of charge. One version can be downloaded from the Demographics 2019 Workshop website at [www.asmda.es](http://www.asmda.es). The program uses the full life tables from the human mortality database to provide the Healthy Life Year Lost estimator  $b_x$  from the general equation form (1):  $b_x = \frac{x\mu_x}{\int_0^x \mu_s ds}$

The Cumulative Mortality  $M_x$  is given by

$$M_x = \int_0^x \mu_s ds \approx \sum_0^x \left( \frac{dx}{lx} \right) \quad (7)$$

Where  $dx$  expresses the death population at age  $x$  in the life tables of the HMD and  $lx$  is the remaining population at age  $x$  in the same life tables. Note that the starting population at age  $x=0$  is set at 100000.

The average mortality  $M_x/x$  is estimated by

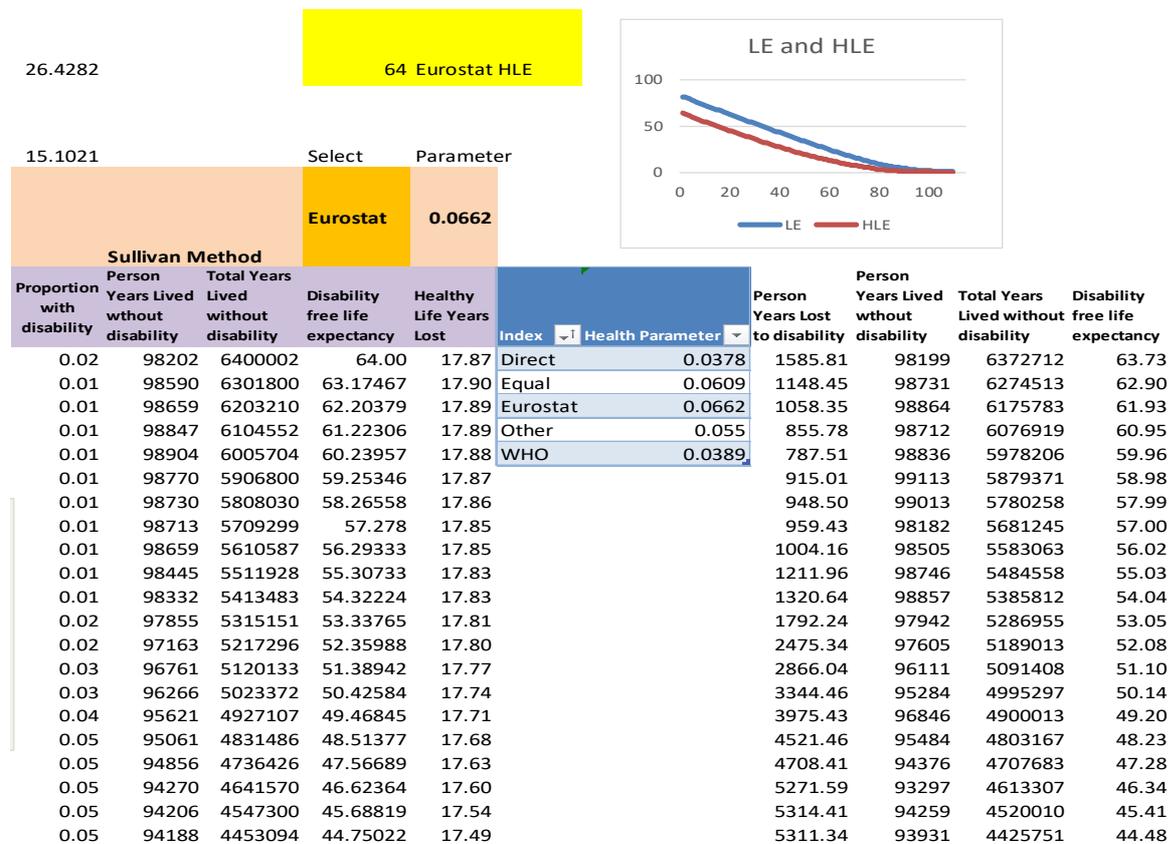
$$\bar{M}_x = \frac{M_x}{x} \approx \frac{\sum_0^x \left( \frac{dx}{lx} \right)}{x}$$



The graph follows a growth process until a high level at 77 years of age and a decline in the remaining lifespan period. In the next column the Healthy Life Year Lost estimator  $b_x$  is provided by dividing the PLYL with the  $l_x$  from the life table. The results are presented in an illustrative graph with the growing trend for  $b_x$  to reach a maximum at 9.71 with a decline at higher ages. This high level can be also estimated by fitting the Weibull model (Weibull 1951).

Another option added in this Excel is the estimates of the World Health Organization from 2000-2016 for Life Expectancy at birth and at 60 years of age for all the member countries whereas information for the Healthy Life Expectancy (HALE) at birth and at 60 years of age is provided for the years 2000, 2005, 2010, 2015 and 2016. We have added a small Table to present comparatively the WHO estimates with our estimates with the direct method. The only needed after copy and paste the life table from the HMD to select the name of the country in L1 and the gender (male, female or both sexes) in L2 in the Excel chart. To avoid mistakes we have used list of the WHO countries with their official names.

**Fig. 3. The extended Life Table for the HLYL estimates with the Sullivan method**



Source: author's calculations

### 3. A further HLE estimate based on the Sullivan method

The classical Sullivan method is a standard device to estimate the healthy life expectancy (see Sullivan, 1971 and Jagger, Van Oyen and Robine, 1999). The simplicity of this method and the possibility to use it as a continuation or extension of the life table gave as the opportunity to add a Sullivan extension to the above extended life table. By this extension we have estimated the Healthy Life Years Lost (HLYL) for all the life span along with the Healthy Life Expectancy (HLE). The main part of the estimate is based on the proportion lived with disability. This is created from the  $bx$  indicator from the previous columns multiplied with a discount Health Parameter. This is estimated directly from the program for our “direct” estimates and for WHO as well. For the Eurostat estimates it is necessary to add the Healthy Life Expectancy at Birth in the appropriate box in the top of Figure 3. Another opportunity is by selecting the “Equal” option which automatically provides an estimate higher than HALE and closer to Eurostat. Everyone of these options should be selected manually in the “Select” and “Parameter” places. The “Other” selection is set by the program user if needed. After selecting the appropriate Health Parameter, the next estimates follow automatically the Sullivan method, first for estimating the person years lived without disability, then the total years lived without disability and finally the Disability Free Life Expectancy. It follows the life years lived with disability column. An alternative method based on the Sullivan system is presented in the columns to the right hand side of Figure 3. The estimates are based on the Person Years Lost to Disability and the direct estimates of the person years lived without disability, the total years lived without disability and the Disability Free Life Expectancy with similar results with the previous approach. The Health Parameters are estimated as follows:

$$\text{Direct Parameter} = \text{sumproduct}(Lx.bx) / (Tx - (LE - HLYL).lx)$$

$$\text{WHO Parameter} = \text{sumproduct}(Lx.bx) / (Tx - (HALE).lx)$$

$$\text{Eurostat Parameter} = \text{sumproduct}(Lx.bx) / (Tx - HLE.lx)$$

The inverse of the Equal Parameter corresponds to the Healthy Life Years Lost, that is:

$$HLYL = 1 / \text{Equal}$$

For the application for Czechia males and females in 2016 the related estimates are included in the next Table.

**TABLE I HLE and HLYL for males and females in Czechia in 2016**

Method	HLE-males	HLYL-males	HLE-females	HLYL-females
Direct	67.7	8.4	71.7	10.2
WHO	67.1	9.2	71.6	10.5
Eurostat	62.7	13.4	64.0	17.9
Equal	60.9	15.2	65.4	16.4
Related estimates from the SKI-6 Program				
SKI-6 Severe+Light	60.8	15.3	61.5	20.4
SKI-6 Severe	68.2	7.9	71.3	10.6
SKI-6 Moderate	68.7	7.4	72.1	9.8

Source: author's calculations

Clearly the estimates for the HLE and the HLYL for males and females are similar for the Direct and the WHO methods. As also is found from the estimates based on the SKI-6 program presented in previous publications both estimates, Direct and WHO, are close to the Severe disability cases. Instead, the estimates provided by the Eurostat (Eurostat, 2019) and Equal are close to Severe and Moderate disability cases. To this end, the Direct and WHO estimates refer to an important part of the life span with the life years lost to disability governing the development of the everyday life.

Our Direct method, further from estimating the Healthy Life Expectancy, has the advantage of providing the Healthy Life Years Lost at every year of age via the  $bx$  parameter. Males and females show similar behavior for the age period 0-70 years of age with the exception of the years from 17-30 where an excess of life years lost to disability appear with the form of a higher  $bx$  for males than for females. After 70 years of age the  $bx$  for females become higher with a maximum level at 88 years of age ( $b=10.21$ ) with a decline for the higher age years. For males the maximum is at 92 years ( $b=8.37$ ) with a decline at higher ages.

## Conclusions and Further Study

We have provided an analytic explanation of the behavior of a parameter  $bx$  related to the healthy life years lost. We have also present an analytic formulation for the observations done along with the development of the appropriate extensions of the classical life tables so that to give a valuable tool for estimating the Healthy Life Years Lost. We have also presented on how the Weibull model

properties expressing the fatigue of materials and especially the cumulative hazard of this model can express the additive process of disabilities and diseases to human population. In this paper we further analytically derive a more general model of survival-mortality in which we estimate a parameter related to the Healthy Life Years Lost (HLYL) and leading to the Weibull model and the corresponding shape parameter as a specific case. We have also demonstrated that the results found for the general HLYL parameter we have proposed provide results similar to those provided by the World Health Organization for the Healthy Life Expectancy (HALE) and the corresponding HLYL estimates. An analytic derivation of the mathematical formulas is presented along with an easy to apply Excel program. A further extension of the Excel program based on the Sullivan method provides estimates of the Healthy Life Expectancy at every year of the lifespan for five different types of estimates that are the Direct, WHO, Eurostat, Equal and Other. The latest versions of this program appear in the Demographics2020 Workshop website at <http://www.smta.net/demographics2020.html> .

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