MIXTURE FUNCTION AS AN APPROPRIATE SMOOTHING OF MORTALITY RATES

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Abstract

Mortality rates represent basic building blocks in demographic and actuarial calculation. From original data of a number of living and a number of deaths, we do not get a smooth mortality rate curve. Our contribution offers a new smoothing method based on so-called mixture functions. Mixture functions represent a special class of weighted averaging functions where weights are determined by continuous, input values dependent, weighting functions. If they are increasing, they form an important class of aggregation functions. Such mixture functions are more flexible than the standard weighted arithmetic mean, and their weighting functions allow one to penalise or reinforce inputs based on their magnitude. This contribution recalls sufficient conditions of the standard monotonicity of mixture functions with selected weighting functions. Based on the mentioned properties, we present an application of the mixture function in smoothing of mortality rates curves.

Key words: force of mortality, death probability, aggregation, mixture function, annuity

JEL Code: C4, C6, J10, J11,

Introduction

Mortality rate is a measure of the number of deaths in a particular population, scaled to the size of that population, per unit of time. There are several different mortality rates used to monitor the level of mortality in populations. So-called crude mortality rates are most commonly used. It counts all deaths, all causes, all ages and both sexes. These data are usually available on the websites of the statistical offices of individual countries, (Statistical Office of the Slovak Republic, 2017), (Czech Statistical Office, 2017).

More complete picture of mortality is given by a life table, which shows the mortality rate separately for each age. Life tables are necessary to give a good estimate of life expectancy.

Very nice approach can be found in (Kaščáková, A., Kubišová, Ľ., Nedelová, G., 2015) and (Kaščáková, A., Nedelová, G., Kubišová, Ľ., 2016). Mortality rate is one of the basic factors considered in setting life insurance premium rates.

Hence, smoothing of mortality rate curves is one of the very important tools for risk management of insurance companies, (Hunt, Blake, 2017), (Yao, Chen, 2016), (Wu, Zeng, 2015) and (Richards, 2008).

Mixture functions represent a special class of weighted averaging functions where weights are determined by continuous, input values dependent, weighting functions, (Beliakov, 2016). If they are increasing, they form an important class of aggregation functions. For more information see (Beliakov, Pradera, Calvo, 2007), (Grabisch, Marichal, Mesiar, Pap, 2009). Such mixture functions are more flexible than the standard weighted arithmetic mean, and their weighting functions allow one to penalise or reinforce inputs based on their magnitude. Because, mixture functions have a lot of very good propeties, and they are also weighted averages with special weights, we wanted to propose new smoothing method just using mixture aggregation. This contribution recalls sufficient conditions of the standard monotonicity of mixture functions with selected weighting functions. Based on the mentioned properties, we present an application of the mixture function in smoothing of mortality rates curves. We have done the whole modelling using the libraries of the R system, (Nakazawa, 2015), (Wickham, 2016) and (R core team, 2017).

1 Preliminaries

At the begining, we recall basic knowledge about aggregation but also about mortality rate smoothing.

1.1. Mixture function as an aggregation function

In this part, we give basic definitions related with our aggregation of mortality rates. Throughout the paper, we give standard monotonicity of mixture functions on the interval $[0, 1]$. The choice of the unit interval is not restrictive. In general, we could study these functions on any arbitrary closed non-empty interval $[a, b] \subset [-\infty, \infty]$, (Špirková, 2008).

Definition 1. (Aggregation function)

A function $F : [0,1]^n \to [0,1]$ is called an n-ary aggregation function if the following conditions hold:

2. [(A1)] F satisfies the boundary conditions $F(0,0,...,0) = 0$ and $F(1,1,...,1) = 1$,
3. [(A2)] F is standard monotone increasing.

Definition 2. (Standard monotonicity)

A function $F:[0,1]^n \to [0,1]$ is monotone increasing if for every $(x_1, x_2,..., x_n)$, $(y_1, y_2,..., y_n) \in [0,1]^n$ such that $x_i \geq y_i$ for every $i = 1,2,..., n$, the inequality $F(x_1, x_2,..., x_n) \geq F(y_1, y_2,..., y_n)$ holds[1].

Definition 3. (Mixture Function) (Špirková, 2008**)**

A function $M_g : [0,1]^n \to [0,1]$ given by

$$M_g(x_1, x_2,..., x_n) = \frac{\sum_{i=1}^{n} g(x_i) \cdot x_i}{\sum_{i=1}^{n} g(x_i)} \tag{1}$$

where $g:[0,1] \to]0,\infty[$ is a continuous weighting function, is called a mixture function.

In general mixture function does not need be monotone. In the next part, we give sufficient conditions of the monotonicity of the mixture function.

Theorem 1.

Mixture function $M_g : [0,1]^n \to [0,1]$ given by (1), is monotone increasing if for an increasing, piecewise differentiable weighting function $g:[0,1] \to]0,\infty[$ at least one from the following conditions is satisfied:

$$g(x) \geq g'(x), \tag{2}$$

$$g(x) \geq g'(x) \cdot (1-x). \tag{3}$$

Remark 1.

Other sufficient conditions of the standard monotonicity of mixture functions and their generalizations can be found, for instance, in (Špirková, 2008).

On the basis of previous conditions, we give the set of coefficients for three types of the weighting functions to be mixture function monotone increasing.

[1] The term "increasing" is understood in a non-strict sense.

Proposition 1.

Let $M_g : [0,1]^n \to [0,1]$ be a mixture function defined by (1) with the weighting function $g_c(x) = cx + 1 - c$, $c \in [0,1]$. Then M_g is monotone increasing with respect to (2) and (3) for

$$c \in [0, 0.5]. \tag{4}$$

Proposition 2.

Let $M_g : [0,1]^n \to [0,1]$ be a mixture function defined by (1) with the weighting function $g_\gamma(x) = 1 + \gamma \cdot x^2$, $\gamma > 0$. Then M_g is monotone increasing:

- with respect to (2) for

$$\gamma \in [0,1], \tag{5}$$

- with respect to (3) for

$$\gamma \in [0,3]. \tag{6}$$

Proposition 3.

Let $M_g : [0,1]^n \to [0,1]$ be a mixture function defined by (1) with the weighting function $g_a(x) = a \cdot \left(\frac{1}{a}\right)^x$, $0 < a < 1$. Then M_g is monotone increasing with respect to (2) and (3) for

$$a \in \left[\frac{1}{e}, 1\right[. \tag{7}$$

Now, we use the mentioned mixture functions on smoothing of mortality rates. Our aim is to compare original smoothing method using moving averages according to the Statistical Office of the Slovak Republic, against our new approach using mixture functions. Our investigation is as follows.

1.2. Mortality rate smoothing using standard method

In determining mortality rates, we start from original data sets of a number of living and a number of deaths. Using a quotient of an actual number of deaths D_x at age x and an actual number of living L_x at age x, we obtain point estimation of the force of mortality x-aged individual in the form $\mu_x = \frac{D_x}{L_x}$.

According to methodological explanations (Statistical Office of the Slovak Republic, 2017) mortality rate q_x at age x, is given by $q_x = \exp\{-\mu_x\}$.

On Figures 1, 2 and 3 (Legend - Statistics) are also illustrated smoothing mortality rates curves which were smoothed using classic moving averages according to methodology of the Statistical Office of the Slovak Republic as follows:

$$q_x = \frac{105 q_x + 90(q_{x-1} + q_{x+1}) + 45(q_{x-2} + q_{x+2}) - 30(q_{x-3} + q_{x+3})}{315}. \tag{8}$$

Moreover, for complete smoothing of the mortality rates curve, the authors needed to apply an interpolation of the second degree, and for higher ages Gompertz-Makeham formula and also King-Hardy method.

2 Mortality rates smoothing using mixture aggregation

As we already mentioned, mixture functions have a lot of very good propeties. We propose new smoothing method just using mixture aggregation. We calculate moving averages using mixture function (1) as follows

$$\hat{q}_x = \frac{\sum_{j=0}^{3} g(q_{x\pm j}) \cdot q_{x\pm j}}{\sum_{j=0}^{3} g(q_{x\pm j})} \tag{9}$$

for $3 \le x \le 75$, consequently with weighting functions $g_c(x) = cx + 1 - c$, $g_\gamma(x) = 1 + \gamma \cdot x^2$ and $g_a(x) = a \cdot \left(\frac{1}{a}\right)^x$, which are mentioned in Propositions 1 to 3.

The main task is to determine which coefficients c, γ or a from the corresponding intervals are the most fitting. We propose to set the most fitting parameters $\hat{c}$, $\hat{\gamma}$ and $\hat{a}$ gradually using formulas

$$\hat{c} = \underbrace{\arg\min}_{(c)} \frac{\sum_{x=3}^{75} \left| q_x - \frac{\sum_{j=0}^{3} (c \cdot q_{x\pm j} + 1 - c) \cdot q_{x\pm j}}{\sum_{j=0}^{3} (c \cdot q_{x\pm j} + 1 - c)} \right| \cdot (L_x + D_x)}{\sum_{x=0}^{100} (L_x + D_x)}, \tag{10}$$

$$\hat{\gamma} = \underbrace{\arg\min}_{(\gamma)} \frac{\sum_{x=3}^{75} \left| q_x - \frac{\sum_{j=0}^{3} \left(1 + \gamma \cdot q_{x \pm j}^2\right) \cdot q_{x \pm j}}{\sum_{j=0}^{3} \left(1 + \gamma \cdot q_{x \pm j}^2\right)} \right| \cdot \left(L_x + D_x\right)}{\sum_{x=0}^{100} \left(L_x + D_x\right)} \tag{11}$$

and

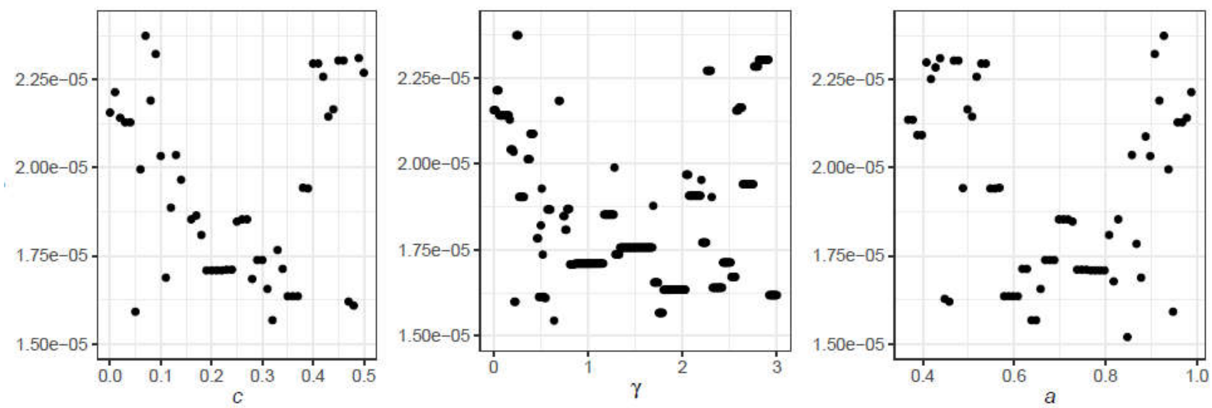
$$\hat{a} = \underbrace{\arg\min}_{(a)} \frac{\sum_{x=3}^{75} \left| q_x - \frac{\sum_{j=0}^{3} \left(a \cdot \left(\frac{1}{a}\right)^{q_{x \pm j}}\right) \cdot q_{x \pm j}}{\sum_{j=0}^{3} \left(a \cdot \left(\frac{1}{a}\right)^{q_{x \pm j}}\right)} \right| \cdot \left(L_x + D_x\right)}{\sum_{x=0}^{100} \left(L_x + D_x\right)}. \tag{12}$$

For higher ages, we applied Gompertz-makeham formula in the R system.

2.1. Statement of the best weighting function

We used the system R on all calculations related to determination of smooth mortality rates curve. On the basis of Figure 1, using formulas (10), (11) and (12), we have chosen fitting parameters $\hat{c} = 0.25$, $\hat{\gamma} = 1.5$ and $\hat{a} = 0.683959$.

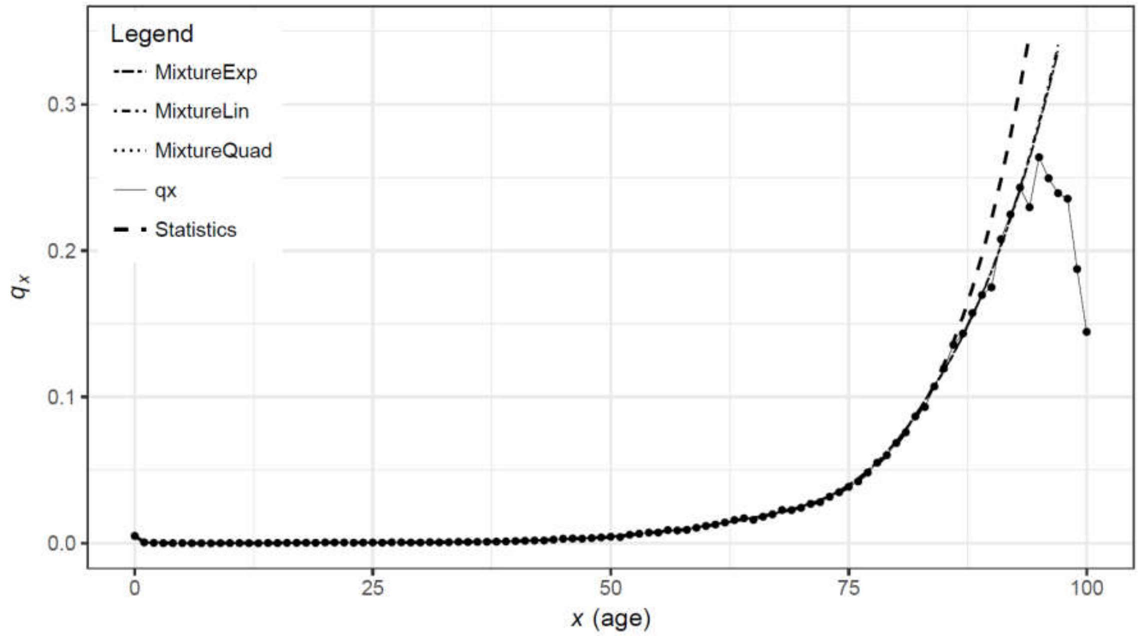
Fig. 1: Determination of the best coefficients on the basis of (10), (11) and (12), (R system)



Source: the authors

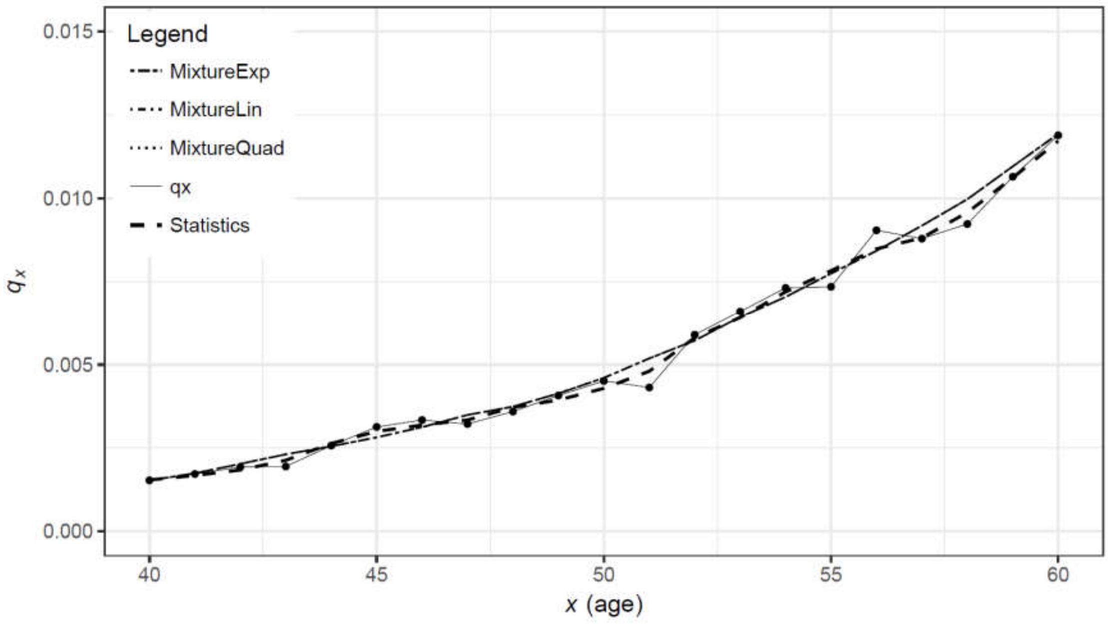
Original curve of q_x (Legend-qx), smoothing mortality rate curves using original method (Legend-Statistics) and also using mixture moving averages (9) and subsequently (10) (Legend-MixtureLin), (11) (Legend-MixtureQuad) and (12) (Legend-MixtureExp) can be seen on the following pictures.

Fig. 2: Mortality rates calculated using standard method and mixture aggregation



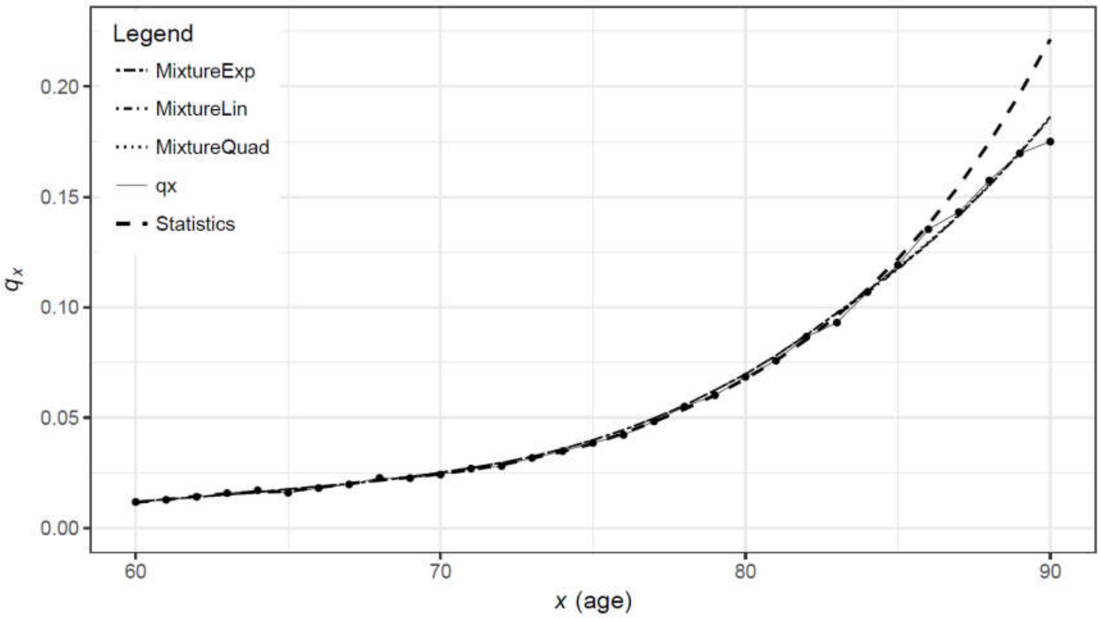
Source: the authors

Fig. 3: Comparison of the mortality rates for lower ages



Source: the authors

Fig. 4: Comparison of the mortality rates for higher ages



Source: the authors

On the basis of our results, ecpecially on the basis of mean square errors, we can state that the best fitting weighting function on aggregation of mortality rates is exponential weighting function with $\hat{a} = 0.6839589$.

Conclusion

In this paper, we proposed a new method on smoothing of mortality rates curve using so-called mixture function. In future, we plan to investigate the impact of both methods on premium for all life insurance products and pension annuities. Moreover, we also plan to study the impact of another weighting functions on mortality rates mixture aggregation, and to determine the most fitting smooth mortality rates curve. All calculations and figures were developed using the statistical R system.

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